Lecture 21

Physics 404

Helium is the only substance that does not solidify at any temperature at atmospheric pressure. Liquid He has a concentration of 51 /nm^3 at 4.2 K, as compared to the quantum concentration of 16 / nm^3, hence $n > n_Q$ and it is a quantum fluid. Quantum mechanics plays a fundamental role in the statistical properties of liquid He.

We will focus today on the ^{4}He isotope. This neutral atom has 2 protons and 2 neutrons in the nucleus, with 2 electrons orbiting. Each of these constituent particles has spin-1/2, but they appear in pairs in the composite object. Hence the ^{4}He atom acts like a composite object of spin 0. It is also chemically inert. Liquid ^{4}He is a fluid of particles that obeys Bose-Einstein statistics with weak interactions. When it is cooled below T_λ= 2.2 K, boiling comes to a stop (see the video). The heat capacity shows a weak divergence at T_λ. Liquid He can flow through narrow constrictions below T_λ but not above. However measurements show that moving objects in He still experience drag.

The Bose-Einstein thermal occupation number $f(E,\mu,\tau) = \frac{1}{e^{(E-\mu)/\tau}-1}$ has the peculiar property that the chemical potential must always remain below the energy values of the states of the system. If not, then $f < 0$, and it makes no sense to have a negative thermal occupation number. Hence the chemical potential is bounded above by the lowest energy eigenstate of the system. We shall take this state (the ground state) to have energy zero, without loss of generality. This means that the chemical potential must always be negative for our considerations here.

We considered the He atoms to be free and non-interacting, but held in a box of side L x L x L. We assume that it is in equilibrium with a reservoir at temperature τ and chemical potential μ. To determine the chemical potential, we enforce the number constraint: $N = \sum_{"S"} f(\varepsilon_S,\mu,\tau)$. We converted this sum to an integral on energy as $N = \int_0^\infty dE\, D(E) f(E,\mu,\tau)$, where the density of states for Bosons is $D(E) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}$, and the Bose-Einstein thermal average occupation number is $f(E,\mu,\tau) = \frac{1}{e^{(E-\mu)/\tau}-1}$. Note that the integral starts at 0 energy, which is the ground state energy. The integral becomes $N = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \tau^{3/2}\frac{\sqrt{\pi}}{2} g(\mu/\tau)$, where $g(\mu/\tau) = \sum_{p=1}^{\infty}\frac{e^{p\mu/\tau}}{p^{3/2}}$. The function $g(\mu/\tau)$ is gently varying from a value of 0 at large and negative values of μ/τ, up to a maximum value of 2.612 at $\frac{\mu}{\tau} = 0$, which is the maximum permissible value of the argument.

The number equation $N = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \tau^{3/2}\frac{\sqrt{\pi}}{2} g(\mu/\tau)$ leads to a crisis as temperature falls. The number N is perhaps on the order of 10^{23}. The right hand side is made up of two temperature dependent quantities, namely τ and $\mu(\tau)$. As temperature drops, the $\tau^{3/2}$ term also drops in magnitude. $\mu(\tau)$ must increase to compensate. However, $g(\mu/\tau)$ can get no larger than 2.612. At that point we can no longer satisfy the number equation, and a crisis ensues. This problem is resolved by noting the fact that in the limit of low temperature a large (i.e. macroscopic) number of particles join the ground state. Our calculation of the particle number includes only the excited state – note that $D(E=0) = 0$, which means we did not include the ground state in our number calculation! This can be corrected by writing: $N = f(E=0,\mu,\tau) + \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\tau^{3/2}\frac{\sqrt{\pi}}{2}g(\mu/\tau)$, and including the thermal average occupation of the ground state explicitly. Including the ground state occupation now balances the equation. One finds that at low temperatures, $\mu \rightarrow 0$ from below, maximizing $g(\mu/\tau)$, but also adding a large number of particles to the ground state. The thermal occupation of the ground state is $f(E=0,\mu,\tau) \rightarrow \frac{1}{e^{-\mu/\tau}-1} = \frac{1}{1-\frac{\mu}{\tau}+\cdots-1} = -\frac{\tau}{\mu}$, for finite but small τ. By tuning the chemical potential very close to zero from below, the system can put an arbitrary number of particles into the ground state at finite temperature τ. Note that the ground state is singled out in this process – no other state enjoys this luxury – see the discussion on pages 201 and 202 of K+K.

The critical temperature for Bose-Einstein condensation can be estimated as the crisis temperature from the original number equation as: $\tau_c = \frac{2\pi\hbar^2}{m}\left(\frac{N/V}{2.612}\right)^{2/3}$. The 'crisis' is that the excited states are no longer populated heavily enough to account for all of the particles. Putting in the numbers for liquid ^{4}He (N/V = 2×10^{28} /m^3) yields $T_c = 3.13\ K$, which is close to the observed lambda-point at 2.2 K. For cold atomic Na (N/V = 10^{20} /m^3) yields $T_c = 1.5\ \mu K$, which is close to the observed Bose-Einstein condensation (BEC) temperature of about $2\ \mu K$ seen in cold atom traps. It is remarkable that an estimate based on a non-interacting gas model can come so close to predicting the actual values of these condensation temperatures.